The Art of Computer Programming Vol. 1 - Some Exercises

An Attempt to solve some of the exercises
from the Book of Donald E. KNUTH

Samuel ORTION

Published by Chameleon Press

samuel.ortion.fr

Introduction

Introduction

This document gathers some random solution to the problem from the *Art of Computer Programming* by Donald E. KNUTH. It should not be considered as a reference, but as a personal attempt to solve some of the exercises, there for the answer should not consider the solution as correct unless he find it so.

The answer are presented with page numbers from the third edition of the AOCP.

1. Basic concepts

1.1 Algorithms

1.2 Mathematical Preliminaries

1.2.1 Mathematical induction

Exercise 4. (p. 18)

Let F_n be the n-th Fibonacci number, defined by the recursion relation

$$\begin{cases} F_0 = 1, \\ F_1 = 1, \\ F_n = F_{n-1} + F_{n-2}, \quad n \geq 2. \end{cases}$$

Let $\phi = \frac{1+\sqrt{5}}{2}$ be the golden ratio.
We have

$$F_n \leq \phi^{n-1} \tag{1.1}$$

$\forall n \in \mathbb{N}$.

Prove that, in addition to Eq. 1.1, $F_n \geq \phi^{n-2}$.

My Answer:
Let $P(n)$ be the assertion that $F_n \geq \phi^{n-2}$.
We proceed by recursion on n.

$$1 + \phi = \phi^2 \tag{1.2}$$

Initiation *For $n = 0$: $\phi^{0-2} = \phi^{-2} = \frac{1}{\phi^2}$. As $\phi^2 > 1$. $\frac{1}{\phi^2} < 1$, so $P(0)$ is true.*
For $n = 1$: $\phi^{1-2} = \phi^{-1} = \frac{1}{\phi}$. As $\phi > 1$, $\frac{1}{\phi} < 1$, so $P(1)$ is true.

Heredity *Let $P(n)$ and $P(n-1)$ be true. We have to prove $P(n+1)$.*

$$\begin{aligned} P(n) \wedge P(n-1) &\Leftrightarrow F_n \geq \phi^{n-2} \wedge F_{n-1} \geq \phi^{n-1-2} \\ &\Leftrightarrow F_n + F_{n-1} \geq \phi^{n-2} + \phi^{n-3} \\ &\Leftrightarrow F_n + F_{n-1} \geq \phi^{n-3}(1+\phi) \\ &\Leftrightarrow F_n + F_{n-1} \geq \phi^{n-3}\phi^2 \qquad \textit{using Eq. 1.2} \\ &\Leftrightarrow F_n + F_{n-1} \geq \phi^{n-1} \\ &\Leftrightarrow F_{n+1} \geq \phi^{n+1-2} \\ &\Leftrightarrow P(n+1) \end{aligned}$$

Conclusion *$P(0)$ and $P(1)$ are true. $P(n) \wedge P(n-1)$ true implies $P(n+1)$ true. So $P(n)$ is true for all $n \in \mathbb{N}$.*

Exercise 5. (p. 19)

A prime number is an integer > 1 that has no exact divisors other than 1 and itself.

Using this definition and mathematical induction, prove that every integer > 1 can be written as a product of prime numbers or is a prime itself.

My Answer: *Let $P(n)$ be the assertion that n can be written as a product of prime numbers or is a prime.*

Proof. Let $N > 1$ be the smallest integer such that $P(N)$ is false.
As $\forall N' < N$, $N' > 1$, $P(N')$ is true. We have either of the following assertions:

- N is a prime number;
- there exists m and n below N such that, $m \times n = N$. As m and n are below N, they satisfies P, so they are either primes or a product of primes. So $mn = N$ is also a product of primes.

We have a contradiction, and $P(N)$ must be true.

∎

Exercise 7. (p. 19)

Formulate and prove by induction a rule for the sums 1^2, 2^2-1^2, $3^2-2^2+1^2$, $4^2-3^2+2^2-1^2$, $5^2-4^2+3^2-2^2+1^2$, etc.

My Answer:
$1^2 = 1$
$2^2 - 1^2 = 3$
$3^2 - 2^2 + 1^2 = 6$
$4^2 - 3^2 + 2^2 - 1^2 = 10$
$5^2 - 4^2 + 3^2 - 2^2 + 1^2 = 15$
We can rewrite the computed sequence with

$$T_n = \sum_{i=0}^{n} (-1)^i (n-i)^2$$

With little help from formal computation, we get

$$T_n = \frac{1}{2}n(n+1) \tag{1.3}$$

Proof. Let us prove the formula.
Let $P(n)$ be the proposition that Eq. 1.3 is correct for T_n.
By recursion on n
Initiation
For $n = 1$, $1^2 = 1$ and $\frac{1}{2}1 \times (1+1) = 1$, so $P(n)$ is true.
Heredity
Suppose $P(n)$ true, let us prove that $P(n+1)$ is also true.

$$\begin{aligned} P(n) &\Leftrightarrow T_n = \frac{1}{2}n(n+1) \\ &\Leftrightarrow \sum_{i=0}^{n}(-1)^i(n-i)^2 = \frac{1}{2}n(n+1) \end{aligned}$$

to be continued... ■

Exercise 8. (p. 19)

(a) Prove the following theorem of Nicomachus by induction:
$1^3 = 1$, $2^3 = 3+5$, $3^3 = 7+9+11$, $4^3 = 13+15+17+19$, etc.
(b) Use this result to prove the remarkable formula $1^3+2^3+\cdots+n^3 = (1+2+\cdots+n)^2$

My Answer:
(a) The theorem states the following:
For all $n \in \mathbb{N}$ we have

$$n^3 = \sum_{i=1}^{n} |n(n-1) - 1 + 2i|$$

Proof. We proceed by recursion on n.
Let $P(n)$ be the proposition "$n^3 = \sum_{i=1}^{n} |n(n-1) - 1 + 2i|$".
Initiation For $n = 1$, $1^3 = 1^2$, so $P(1)$ is true.
Heredity Suppose $P(n)$ true, let us prove that $P(n+1)$ is also true.

$$P(n) \Leftrightarrow n^3 = \sum_{i=1}^{n} n(n-1) - 1 + 2i$$

■

to be continued...
(b)

Theorem 1.1 — Formula of Nicomachus. The sum of the cubes of the first n natural numbers is equal to the square of the sum of the first n natural numbers.

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2 \tag{1.4}$$

Proof. Let $P(n)$ be the proposition "$\sum_{k=1}^{n} k^3 = (\sum_{k=1}^{n} k)^2$".
$P(n) \Longleftrightarrow \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$ because $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.
We want to prove that: $\sum_{k=1}^{n+1} k^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2$

$$\begin{aligned}\sum_{k=1}^{n+1} k^3 &= \frac{n^2(n+1)^2}{2^2} + (n+1)^3 \\ &= (n+1)^2\left(\frac{n^2}{4} + n + 1\right) \\ &= (n+1)^2 \cdot \frac{n^2+4n+4}{4} \\ &= \frac{(n+1)^2(n+2)^2}{2^2} \\ &\Updownarrow \\ &P(n+1)\end{aligned}$$

Conclusion
$P(1)$ is true because $1^3 = 1$ and $\left(\frac{1(1+1)}{2}\right)^2 = 1$.
$P(n)$ true implies $P(n+1)$ true, so $P(n)$ is true for all $n \in \mathbb{N}_+$. ■

Exercise 9. (p. 19)

Prove by induction that if $0 < a < 1$ then $(1-a)^n \geq 1 - na$.

My Answer: *We proceed by recursion on n.*
Let $P(n)$ be the proposition "$(1-a)^n \geq 1 - na$".
Initiation For $n = 0$,
$(1-a)^n = 1 = 1 - na$, so $P(0)$ is true.
Heredity Suppose $P(n)$ true, let us prove that $P(n+1)$ is also true.
We want to prove that $(1-a)^{n+1} \geq 1 - (n+1)a$.

$$\begin{aligned}P(n) \Leftrightarrow (1-a)^n &\geq 1 - na \\ (1-a)^n(1-a) &\geq (1-na)(1-a) \qquad 1 - a > 0 \\ (1-a)^{n+1} &\geq 1 - na - a + na \\ &\geq 1 - (n+1)a - a \\ &\geq 1 - (n+1)a \qquad \textit{because } a > 0 \qquad \Rightarrow P(n+1)\end{aligned}$$

Conclusion $P(1)$ is true.
$P(n)$ true implies $P(n+1)$ true, so $P(n)$ is true for all $n \in \mathbb{N}_+$.

1.2.2 Numbers, Powers and Logarithms

Exercise 1. (p. 25)

What is the smallest positive rational number?

My Answer: *The smallest positive rational number is 0.*

Exercise 2. (p. 25)

Is $1 + 0.23999999999...$ a decimal expansion?

My Answer: *0.23999999999... is not a decimal number as it has an infinite decimal expansion.*
The decimal expansion of $1 + 0.23999999999...$ *is in the form*

$$1 \cdot 10^0 + 2 \cdot 10^{-1} + 3 \cdot 10^{-2} + 9 \cdot 10^{-3} + \sum_{i=1}^{\infty} 9 \cdot 10^{-3-i}$$

Nota bene: This surely does not answer the question...

Exercise 3. (p. 25)

What is $(-3)^{-3}$?

My Answer:

$$(-3)^{-3} = \frac{1}{(-3)^3} = \frac{1}{(-3)(-3)(-3)} = -\frac{1}{27}$$

Exercise 4. (p. 25)

What is $(0.125)^{-2/3}$?

$$b^{p/q} = \sqrt[q]{b^p} \tag{1.5}$$

My Answer:

$$\begin{aligned} 0.125^{-2/3} &= \left(\frac{1}{8}\right)^{-2/3} \\ &= 8^{2/3} \\ &= \sqrt[3]{8^2} \\ &= \sqrt[3]{64} \\ &= 4 \end{aligned}$$

Exercise 5. (p. 25)

Defining real numbers in terms of binary expansion.

My Answer:

$$x = \sum_{i=0}^{\infty} a_i \cdot 2^{-i}$$

Exercise 6. (p. 25)

Let $x = m + 0.d_1d_2\ldots$ and $y = n + 0.e_1e_2\ldots$. Give a rule for determining whether $x = y$, $x < y$, $x > y$ based on the decimal representation.

My Answer:

Algorithm 1 Greater, Lower or Equal Decimal representation

procedure LowerGreaterEqual(X, Y) ▷ *X and Y are arrays of decimal of numbers x and y respectively.*
    $i \leftarrow 0$
    while $i < X.length$ and $i < Y.length$ **do**
        if $X[i] == Y[i]$ **then**
            $i \leftarrow i + 1$
        else if $X[i] > Y[i]$ **then**
            return `GREATER`
        else
            return `LOWER`
    if $X.length > Y.length$ **then**
        return `GREATER`
    else if $X.length < Y.length$ **then**
        return `LOWER`
    else
        return `EQUALS`

Exercise 7. (p. 25)

Given that x and y are integers. *Prove* the laws of exponents, from definition given by Eq. 1.7 from Eq. 1.6.

$$b^0 = 1, \quad b^n = b^{n-1}b \quad \text{if } n > 0, \quad b^n = b^{n+1}/b \quad \text{if } n < 0. \tag{1.6}$$

$$b^{x+y} = b^x b^y, \quad (b^x)^y = b^{xy}, \tag{1.7}$$

My Answer: *We will prove this in $\mathbb{R}$ Let us recall the following formula:*

$$b^x = e^{x \ln b} \tag{1.8}$$

and

$$\exp(x + y) = \exp(x)\exp(y) \tag{1.9}$$

Thus it comes:

$$\begin{aligned} b^{x+y} &= e^{(x+y)\ln b} \\ &= e^{x \ln b + y \ln b} \\ &= e^{x \ln b} e^{y \ln b} \\ &= b^x b^y \end{aligned}$$

Similarly for $(a^x)^y$:

$$\begin{aligned} (a^x)^y &= e^{y \ln(a^x)} \\ &= e^{xy \ln a} \\ &= a^{xy} \end{aligned}$$

Conclusion

Bibliography

Articles

Books

Contents