The Art of Computer Programming Vol. 1 - Some Exercises

An Attempt to solve some of the exercises
from the Book of Donald E. KNUTH

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Introduction

Introduction

1. Basic concepts

1.1 Algorithms

1.2 Mathematical Preliminaries

1.2.1 Mathematical induction

Exercise 4. (p. 18)

Let F_n be the n-th Fibonacci number, defined by the recursion relation

$$\begin{cases} F_0 = 1, \\ F_1 = 1, \\ F_n = F_{n-1} + F_{n-2}, \quad n \geq 2. \end{cases}$$

Let $\phi = \frac{1+\sqrt{5}}{2}$ be the golden ratio.
We have

$$F_n \leq \phi^{n-1} \tag{1.1}$$

$\forall n \in \mathbb{N}$.
Prove that, in addition to Eq. 1.1, $F_n \geq \phi^{n-2}$.

My Answer:

Let $P(n)$ be the assertion that $F_n \geq \phi^{n-2}$.
We proceed by recursion on n.

$$1 + \phi = \phi^2 \tag{1.2}$$

Initiation For $n = 0$: $\phi^{0-2} = \phi^{-2} = \frac{1}{\phi^2}$. As $\phi^2 > 1$. $\frac{1}{\phi^2} < 1$, so $P(0)$ is true.
For $n = 1$: $\phi^{1-2} = \phi^{-1} = \frac{1}{\phi}$. As $\phi > 1$, $\frac{1}{\phi} < 1$, so $P(1)$ is true.
Heredity Let $P(n)$ and $P(n-1)$ be true. We have to prove $P(n+1)$.

$$\begin{aligned} P(n) \wedge P(n-1) &\Leftrightarrow F_n \geq \phi^{n-2} \wedge F_{n-1} \geq \phi^{n-1-2} \\ &\Leftrightarrow F_n + F_{n-1} \geq \phi^{n-2} + \phi^{n-3} \\ &\Leftrightarrow F_n + F_{n-1} \geq \phi^{n-3}(1+\phi) \\ &\Leftrightarrow F_n + F_{n-1} \geq \phi^{n-3}\phi^2 \qquad \textit{using Eq. 1.2} \\ &\Leftrightarrow F_n + F_{n-1} \geq \phi^{n-1} \\ &\Leftrightarrow F_{n+1} \geq \phi^{n+1-2} \\ &\Leftrightarrow P(n+1) \end{aligned}$$

Conclusion $P(0)$ and $P(1)$ are true. $P(n) \wedge P(n-1)$ true implies $P(n+1)$ true. So $P(n)$ is true for all $n \in \mathbb{N}$.

Exercise 5. (p. 19)

A prime number is an integer > 1 that has no exact divisors other than 1 and itself. Using this definition and mathematical induction, prove that every integer > 1 can be written as a product of prime numbers or is a prime itself.

My Answer: *Let $P(n)$ be the assertion that n can be written as a product of prime numbers or is a prime.*

Proof. Let $N > 1$ be the smallest integer such that $P(N)$ is false.
As $\forall N' < N$, $N' > 1$, $P(N')$ is true. We have either of the following assertions:

- N is a prime number;
- there exists m and n below N such that, $m \times n = N$. As m and n are below N, they satisfies P, so they are either primes or a product of primes. So $mn = N$ is also a product of primes.

We have a contradiction, and $P(N)$ must be true.

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Exercise 7. (p. 19)

Formulate and prove by induction a rule for the sums 1^2, $2^2 - 1^2$, $3^2 - 2^2 + 1^2$, $4^2 - 3^2 + 2^2 - 1^2$, $5^2 - 4^2 + 3^2 - 2^2 + 1^2$, etc.

My Answer:

$1^2 = 1$
$2^2 - 1^2 = 3$
$3^2 - 2^2 + 1^2 = 6$
$4^2 - 3^2 + 2^2 - 1^2 = 10$
$5^2 - 4^2 + 3^2 - 2^2 + 1^2 = 15$

We can rewrite the computed sequence with

$$T_n = \sum_{i=0}^{n} (-1)^i (n-i)^2$$

With little help from formal computation, we get

$$T_n = \frac{1}{2} n(n+1) \tag{1.3}$$

Proof. Let us prove the formula.
Let $P(n)$ be the proposition that Eq. 1.3 is correct for T_n.
By recursion on n

Initiation
For $n = 1$, $1^2 = 1$ and $\frac{1}{2} 1 \times (1 + 1) = 1$, so $P(n)$ is true.

Heredity
Suppose $P(n)$ true, let us prove that $P(n + 1)$ is also true.

$$\begin{aligned} P(n) &\Leftrightarrow T_n = \frac{1}{2} n(n+1) \\ &\Leftrightarrow \sum_{i=0}^{n} (-1)^i (n-i)^2 = \frac{1}{2} n(n+1) \end{aligned}$$

to be continued...

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Exercise 8. (p. 19)

(a) Prove the following theorem of Nicomachus by induction:
$1^3 = 1$, $2^3 = 3 + 5$, $3^3 = 7 + 9 + 11$, $4^3 = 13 + 15 + 17 + 19$, etc.
(b) Use this result to prove the remarkable formula $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$

My Answer:

(a) The theorem states the following:
For all $n \in \mathbb{N}$ we have

$$n^3 = \sum_{i=1}^{n} |n(n-1) - 1 + 2i|$$

Proof. We proceed by recursion on n.
Let $P(n)$ be the proposition "$n^3 = \sum_{i=1}^{n} |n(n-1) - 1 + 2i|$".
Initiation For $n = 1$, $1^3 = 1^2$, so $P(1)$ is true.
Heredity Suppose $P(n)$ true, let us prove that $P(n + 1)$ is also true.

$$P(n) \Leftrightarrow n^3 = \sum_{i=1}^{n} n(n-1) - 1 + 2i$$

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to be continued...

(b)

Theorem 1.1 — Formula of Nicomachus. The sum of the cubes of the first n natural numbers is equal to the square of the sum of the first n natural numbers.

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2 \tag{1.4}$$

Proof. Let $P(n)$ be the proposition "$\sum_{k=1}^{n} k^3 = (\sum_{k=1}^{n} k)^2$".
$P(n) \iff \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$ because $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.
We want to prove that: $\sum_{k=1}^{n+1} k^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2$

$$\begin{aligned}
\sum_{k=1}^{n+1} k^3 &= \frac{n^2(n+1)^2}{2^2} + (n+1)^3 \\
&= (n+1)^2 \left(\frac{n^2}{4} + n + 1\right) \\
&= (n+1)^2 \cdot \frac{n^2 + 4n + 4}{4} \\
&= \frac{(n+1)^2(n+2)^2}{2^2} \\
&\Updownarrow \\
&P(n+1)
\end{aligned}$$

Conclusion
$P(1)$ is true because $1^3 = 1$ and $\left(\frac{1(1+1)}{2}\right)^2 = 1$.
$P(n)$ true implies $P(n + 1)$ true, so $P(n)$ is true for all $n \in \mathbb{N}_+$. ■

Exercise 9. (p. 19)

Prove by induction that if $0 < a < 1$ then $(1-a)^n \geq 1 - na$.
My Answer: *We proceed by recursion on n.*
Let $P(n)$ be the proposition "$(1-a)^n \geq 1 - na$".
Initiation For $n = 0$,
$(1-a)^n = 1 = 1 - na$, so $P(0)$ is true.
Heredity Suppose $P(n)$ true, let us prove that $P(n+1)$ is also true.
We want to prove that $(1-a)^{n+1} \geq 1 - (n+1)a$.

$$\begin{aligned} P(n) \Leftrightarrow (1-a)^n &\geq 1 - na \\ (1-a)^n(1-a) &\geq (1-na)(1-a) \qquad 1-a>0 \\ (1-a)^{n+1} &\geq 1 - na - a + na \\ &\geq 1 - (n+1)a - a \\ &\geq 1 - (n+1)a \qquad \textit{because } a>0 \qquad \Rightarrow P(n+1) \end{aligned}$$

Conclusion $P(1)$ is true.
$P(n)$ true implies $P(n+1)$ true, so $P(n)$ is true for all $n \in \mathbb{N}_+$.

Conclusion

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